Hann-Banach-Arveson extension theorem and Hyper-rigidity

Anilesh Mohari

The Institute of Mathematical Sciences,

CIT Campus, Tharamani, Chennai-600113

E-mail:anilesh@imsc.res.in

Date: 20th May, 2013

Abstract

Let $C^*(\mathcal{S})$ be the C^* algebra generated by an operator system $\mathcal{S}$ i.e. a unital $*$-closed subspace of a unital C^* algebra $\mathcal{A}$. We consider here separable operator systems and prove that any complete order isomorphism $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ between two such operator systems has a unique extension to a C^*-isomorphism $\mathcal{I} : C^*(\mathcal{S}) \to C^*(\mathcal{S}')$. As an application of this result we settle Arveson's conjecture: any separable operator system $\mathcal{S}$ is hyper-rigid for $C^*(\mathcal{S})$ if and only if every irreducible representation of $C^*(\mathcal{S})$ is a boundary representation of the operator system $\mathcal{S}$.

1 Introduction:

Let $C(\Omega)$ be the commutative C^*-algebra of countinuous complex valued functions on a compact Hausdorff space Ω. For two such spaces Ω and Ω' and a continuous map $\gamma : \Omega' \to \Omega$ we define endomorphism $\Gamma : C(\Omega) \to C(\Omega')$ defined by

$$\Gamma(\psi)(\omega') = \psi \circ \gamma(\omega') \tag{1.1}$$

In case γ is an one to one and onto map, then Γ is an automorphism. A well known theorem, due to M. Stone [Sto2], says that any auto-morphism $\Gamma : C(\Omega) \to C(\Omega')$ determines uniquely a continuous one to one and onto map $\gamma : \Omega' \to \Omega$ such that (1.1) holds. A theorem of R.V. Kadison [Ka1] says that same is true if Γ is an *order isomorphism* i.e. a linear one to one and onto map such that both Γ and Γ^{-1} are non-negative maps i.e. the map takes non-negative functions to non-negative functions. A subspace $\mathcal{F} \subseteq C(\Omega)$ is called a *functions system* if $\mathcal{F}$ contain constant functions and $\mathcal{F}$ is closed under conjugation i.e. $\psi \in \mathcal{F}$ if and only if $\bar{\psi}(\omega) = \bar{\psi(\omega)}$. Another theorem of M. Stone [Sto1] also says that closed algebra generated by a function system $\mathcal{F}$ is equal to $C(\Omega)$ if and only if $\mathcal{F}$ separates points in Ω. In particular this result also says that an automorphism Γ is determined unique by it's action on a separating point function system $\mathcal{F}$. One valid question that one may ask: What can be said about an order isomorphism map between two separating points function systems $\mathcal{F}$ and $\mathcal{F}'$?

In this paper we address this problem with affirmative answer in a more general framework of *operator systems* of a C^*-algebra replacing function systems of $C(\Omega)$ and then make use of these results to answer Arveson's conjecture on hyper-rigidity of an operator system which we describe in details after developing our mathematical framework of C^*-algebras and their associated maps.

A vector subspace $\mathcal{S}$ of a unital C^* algebra $\mathcal{A}$ is called self-adjoin if $x^* \in \mathcal{S}$ whenever $x \in \mathcal{S}$. A self-adjoint unital subspace $\mathcal{S}$ of $\mathcal{A}$ is called *operator system* [Ar1,Pa3]. Let $\mathcal{S}_+$ be the cone of positive elements in $\mathcal{S}$ and $C^*(\mathcal{S})$ to be the C^*-

algebra generated by $\mathcal{S}$. An unital linear map $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ between two operator systems $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{S}' \subseteq \mathcal{A}'$ is called positive if $\mathcal{I}(\mathcal{S}_+) \subseteq \mathcal{S}'_+$ and is called completely positive (CP) if for each $k \geq 1$, $\mathcal{I}^{(k)} = \mathcal{I} \otimes I_k : M_k(\mathcal{S}) \to M_k(\mathcal{S}')$ defined by $\mathcal{I} \otimes T_k((x^i_j)) = ((\mathcal{I}(x^i_j)))$ is positive i.e. $\mathcal{I}^{(k)}(M_k(\mathcal{S})_+) \subseteq M_k(\mathcal{S}')_+$. Two operator systems $\mathcal{S}$ and $\mathcal{S}'$ are called order isomorphic if there exists $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$, a positive unital one to one and onto linear map such that it's inverse is also positive. Two operator systems $\mathcal{S}$ and $\mathcal{S}'$ are called completely order isomorphic if there exists a unital completely positive (UCP) $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ one to one and onto map such that it's inverse is also completely positive. We have cited standard text [Pa3] on operator systems on several instants in the text omitting often details as we maintained same terminology and hopefully also same notations.

In a celebrated paper [Ka1] R.V. Kadison proved that an order isomorphism between two C^* algebras is a sum of a morphism and an anti-morphism i.e. for two arbitrary C^*-algebras $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ an order-isomorphism $\mathcal{I} : \mathcal{A} \to \mathcal{B}$ is a disjoint sum of a morphism and an anti-morphism i.e. there exists a projection $e \in \mathcal{B}'' \bigcap \mathcal{B}'$ i.e. center of $\mathcal{B}$ such that $x \to \mathcal{I}(x)e$ is morphism ($*$-linear and multiplicative) and $x \to \mathcal{I}(x)(I-e)$ is an anti-morphism ($*$-linear and anti-multiplicative). Thus when $\mathcal{B}$ is a factor an order-isomorphism is either an isomorphism or anti-isomorphism. It is a simple observation that anti-morphism part in the decomposition will be absent [ER] if we also demand an order isomorphism to be a CP map. For details we refer to Corollary 5.2.3 in [ER]. In particular [Fa] we have a norm preserving CP unital linear map $\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, i.e. $||\tau(x)|| = ||x||$ for all $x \in M_n(\mathbb{C})$ if and only if $\tau(x) = uxu^*$ for some unitary $u \in M_n(\mathbb{C})$. Thus a complete order isomorphism on C^* algebras is an C^*-isomorphism which we call in short isomorphism.

It is natural to look for a generalization of R.V. Kadison's theorem for operator systems with the role of order isomorphism replaced by complete order isomorphism between two operator systems. This problem got it's attention ever since William Arveson introduced dilation theory [Ar1].

THEOREM 1.1: Let $\mathcal{S}, \mathcal{S}'$ be two separable operator systems of unital C^* algebras and $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ be a unital complete order isomorphism. Then there exists a unique complete order isomorphism $\mathcal{I} : C^*(\mathcal{S}) \to C^*(\mathcal{S}')$ extending $\mathcal{I}$. Further same statement is valid for non-separable operator systems as well.

First we prove Theorem 1.1 with additional hypothesis on complete order-isomorphism map $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ namely we will assume that it takes a faithful state ϕ on $C^*(\mathcal{S})$ to another faithful state ϕ' on $C^*(\mathcal{S})$ i.e. $\phi' = \phi \circ \mathcal{I}$ on $\mathcal{S}$. To prove this weaker version of Theorem 1.1, we adopt Arveson's ideas appeared in [Ar1] which uses an one to one and onto correspondence between the category of unital CP maps on an operator system and states on an amplified operators systems. The following theorem of independent interest is our main technical theorem.

THEOREM 1.2: Let $\mathcal{S}$ be a separable operator system in a unital C^* algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and closed in weak* topology of $\mathcal{B}(\mathcal{H})$. Let $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H}')$ be a unital injective weak* continuous CP map. Let ϕ_0 be a faithful normal state on $\mathcal{B}(\mathcal{H}')$. Then the following holds:
(a) There exists a faithful state ϕ on $\mathcal{A}$ such that $\phi(x) = \phi_0(\tau(x))$ for all $x \in \mathcal{S}$;
(b) For each faithful state ϕ satisfying (a), τ has a CP extension $\eta : \mathcal{A} \to \mathcal{B}(\mathcal{H}')$ of $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H}')$ such that $\phi_0 \eta = \phi$ on $\mathcal{A}$.

Theorem 1.2 is the basic ingredient used to find a proof for Theorem 1.1 with additional hypothesis on intertwining faithful states. The condition of having intertwining faithful states can always be fulfilled when operator systems are separable. Finally separable assumption on operator systems can also be removed by considering the class of separable operator systems contained in $\mathcal{S}$ and $\mathcal{S}'$ respectively. Then we can use transfinite induction in order to get a global isomorphisms from $C^*(\mathcal{S})$ to $C^*(\mathcal{S}')$. Thus we arrive at the full generality of Theorem 1.1. Proof of Theorem 1.1 and Theorem 1.2 will be given in section 3.

In different contexts with different related objectives and motivations, operator

systems are studied (see papers and references there in [Ag],[Da] for model theory, [Ar4],[Ar5],[MuS] for non -commutative Choquet's boundary theory, [Bl],[BlM] for multiplier algebra and Morita equivalence, [Pa1],[Pa2],[Ha] for Kadison conjecture on similarity, [Pi] for injective property of a C^* algebra and Kadison' similarity problem) for the last few decades ever since William Arveson's celebrated work [Ar1] gave the foundation for operator systems. We also wish to draw reader's attention to some interesting related problems reviewed in a recent elegant exposition by F. Douglas on unitary problem [Fa] on similarity.

In the following we include our prime application by proving Arveson's conjecture on *hyper-rigidity* which would be the basis for a non-commutative potential theory. We defer other non trivial applications of Theorem 1.1. A complete classification problem up to cocycle conjugacy for extremal elements in the set of UCP maps on $M_n(C)$ [Ch] will be investigated in a forth coming paper [Mo].

We first recall quickly Shilov and Choquet boundary of a function system. Let $A = C(\Omega)$ be a unital commutative Banach algebra of continuous functions on a compact Hausdorff space. A closed subset F of Ω is called a boundary of A or Ω if

$$\max_{\omega \in F}|x(\omega)| = \max_{\omega \in \Omega}|x(\omega)|$$

i.e. an abstract maximum modulus principle holds. We use symbol ∂A to denote the set of boundary sets of A. The closed set $K = \bigcap_{F \in \partial A} F$ is also a boundary by a theorem of Georgiy Shilov and K is called the Shilov boundary of A. Thus one may also say that Shilov boundary is the minimal closed set where abstract maximal modulus principle holds good for A. Thus $\mathcal{M}_A$ is the algebra of continuous functions on the Shilov boundary K of Ω with the imbedding $i : f \to f_{|}K$. Instead of A, we can define similarly Shilov boundary $K_{\mathcal{S}}$ of a function system $\mathcal{S} \subseteq A$ if $\mathcal{S}$ separates two points of Ω and contains constant functions. $K_{\mathcal{S}}$ is the smallest closed set in Ω on which every function in $\mathcal{S}$ achieves its norm. $\mathcal{M}_{\mathcal{S}}$ is the algebra of continuous functions on K_s. In particular it says that $K_{\mathcal{S}}$ is the closure of Choquet boundary of Ω relative to $\mathcal{S}$.

We fix an operator system $\mathcal{S}$ and set $\mathcal{A} = C^*(\mathcal{S})$. We recall now abstract characterization of C^*-envelope algebra as described in [Bl] inspired by [Ar1],[Ha] and [DM]. The C^*-envelope algebra of an operator system $\mathcal{S}$ consists of a C^* algebra $\mathcal{M}_{\mathcal{S}}$ and a completely isometric unital imbedding $i : \mathcal{S} \to \mathcal{M}_{\mathcal{S}}$ such that $\mathcal{M}_{\mathcal{S}} = C^*(i(\mathcal{S}))$ with the following universal factorization property: If $j : \mathcal{S} \to \mathcal{B} = C^*(j(\mathcal{S}))$ is another such an imbedding then there is a $*$-homomorphism $\pi : \mathcal{B} \to \mathcal{M}_{\mathcal{S}}$ so that $i = j\pi$. More generally given an (abstract) operator algebra A, the C^*-envelope, $\mathcal{M}_A$, is the (essentially unique) smallest C^*-algebra among those C^*-algebras C that can be generated by a completely isometric representation of A. Thus, the C^*-envelope can be seen as the non-commutative analogue of the Shilov boundary.

DEFINITION 1.3: A UCP map $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is said to have *unique extension property* if

(i) τ has a unique CP extension $\tilde{\tau} : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$;

(ii) $\tilde{\tau}$ is a representation of $C^*(\mathcal{S})$ on $\mathcal{H}$.

If $\mathcal{H}$ is separable we say τ is *separably acting.*

DEFINITION 1.4: An irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex separable Hilbert space, is called a *boundary representation* if $\pi_{|\mathcal{S}}$ has unique extension property i.e. for any UCP map $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\tau = \pi$ on $\mathcal{S}$ then $\tau = \pi$ on $\mathcal{A}$.

The C^*-envelope algebra was introduced by W. B. Arveson in [Ar1] and also [Ar2]. He proved its existence provided there are enough boundary representations $\partial\mathcal{S}$: For each $n \geq 1$ the following holds

$$||(x^i_j)|| = \sum_{\pi \in \partial\mathcal{S}} ||((\pi(x^i_j)))||$$

for all $x = (x^i_j) \in M_n(\mathcal{S})$. One such realization of $\mathcal{M}_{\mathcal{S}}$ in his work in [Ar1,Ar2] goes as follows. The largest closed two sided ideal $K_{\mathcal{S}}$ of $\mathcal{A}$ such that the quotient map $x \to [x] = x + K_{\mathcal{S}}$ is completely isometric on $\mathcal{S}$ where $K_{\mathcal{S}} = \{\bigcap\{ker(\pi) : \pi \in \partial\mathcal{S}\}$ where $\partial\mathcal{S}$ is the set of *boundary representation* of $\mathcal{S}$. The quotient C^*-algebra is

the C^*-envelope of $\mathcal{S}$. In the commutative setting, these boundary representations correspond to points in the Choquet boundary.

These results on existence of Shilov's boundary have some influence on William Arveson's reformulation on his non-commutative Choquet boundary theory and it's relation with Shilov's boundary. Question that came as central when those two boundaries are equal in the non-commutative set up. In his paper [Ar4] this problem where reformulated into a number of equivalent criteria in terms of non-commutative version of classical approximation theorem of P. P. Korovkin [Ko]. We refer Theorem 2.1 in [Ar4] for details. We adopt criteria (c) given in Theorem 2.1 in [Ar4] as our definition of hyper-rigidity in the following definition.

DEFINITION 1.5: An operator system $\mathcal{S}$ of a unital C^*-algebra is called *hyper-rigid* for $\mathcal{A}$ if $\mathcal{S}$ generates $\mathcal{A}$ i.e. the minimal sub-algebra $C^*(\mathcal{S})$ of $\mathcal{A}$ that contains $\mathcal{S}$ is $\mathcal{A}$ itself and every representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ has a unique CP extension i.e. only UCP map $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that satisfies $\tau_{|}\mathcal{S} = \pi_{|}\mathcal{S}$ is $\tau = \pi$ itself.

Theorem 2.2.5 in [Ar1] gives a weaker version for Theorem 1.1 under additional hypothesis namely trivial 'Shilov boundary ideals K_s and $K_{s'}$ for $\mathcal{S}$ and $\mathcal{S}'$ respectively. In Corollary 4.2 in [Ar4] he has also proved for separable operator systems 'hyper-rigidity' implies trivial Shilov's ideal. Further he has also given an explicit example of an operator system which fails hyper-rigidity property (Theorem 3.1 in [Ar4]). On the other hand now Theorem 1.1 shows that hyper-rigidity has no direct relation with complete order isomorphism lifting property from operator systems to their C^*-algebras. However our main result Theorem 1.1 finds it's place in the proof given here for an criteria for 'hyper-rigidity'.

Now onwards our attention is restricted to separable operator systems $\mathcal{S}$. The main reason behind this assumption lies with the fact that it guarantees existence of a faithful state and so we may embed $\mathcal{A} = C^*(\mathcal{S})$ faithfully into the algebra of

bounded operators on a complex separable Hilbert space via GNS construction [BR]. William Arveson proved in [Ar4] that every irreducible representation of $C^*(\mathcal{S})$ is a boundary representation of $\mathcal{S}$ if $\mathcal{S}$ is hyper-rigid. In such a situation C^*-envelope $\mathcal{M}_{\mathcal{S}}$ is trivial. In his paper he then asked whether converse is true? This has influenced us to state and prove the following theorem which he has stated as Conjecture 4.3 in [Ar4].

THEOREM 1.6: Let $\mathcal{S}$ be a separable operator system of $\mathcal{A} = C^*(\mathcal{S})$. Then following statements are equivalent:

(a) $\mathcal{S}$ is hyper-rigid for $C^*(\mathcal{S})$.

(b) Every irreducible representation of $C^*(\mathcal{S})$ is a boundary representation for $\mathcal{S}$;

Further in such a case

(c) $\mathcal{M}_{\mathcal{S}} = C^*(\mathcal{S})$ and K is trivial.

Theorem 1.1 in particluar says that for two bounded operators x, y on a Hilbert space the map

$$\lambda I + \mu x \to \lambda I + \mu y,\ \lambda, \mu \in C$$

is completely isometric if and only if the map extends to a C^* isomorphism bewteen $\mathcal{A}(x)$ and $\mathcal{A}(y)$ where we used symbol $\mathcal{A}(x)$ for the C^*-subalgebra generated by x. Further the isomorphism takes one faithful state on $\mathcal{A}(x)$ to another faithful state as C^* algebras $\mathcal{A}(x)$ and $\mathcal{A}(y)$ are separable (for details see section 2). Thus via GNS representation we can emmbed faithfully $\pi_x : \mathcal{A}(x) \to \mathcal{B}(\mathcal{H}_x)$ and find an unitary operator $U : \mathcal{H}_x \to \mathcal{H}_y$ such that

$$U\pi_x(x)U^* = \pi_y(y)$$

In case x, y are irreducible compact operators on $\mathcal{H}$, $\mathcal{A}_x = \mathcal{A}_y = \mathcal{K}(\mathcal{H})$ i.e. equal to C^*-algebra of all compact operators on $\mathcal{H}$, then isomorphism is intertwined by a unique unitary operator u on $\mathcal{H}$ i.e.

$$uxu^* = y$$

This result with irreducible compact operators x, y is well known [Ar1]. In case x, y are bounded operators on a Hilbert space, a general mathematical problem of interest when can we expect the natural complete isomorphism map which takes x to y are inter-twined an unitary operator u on $\mathcal{H}$?

Now we also discuss briefly significance of Theorem 1.1 in classical set up described at the starting paragraph. Theorem 1.1 says now that any order isomorphism map between two seprating points function systems $\mathcal{F}$ and $\mathcal{F}'$ has a unique extension to an automorphism between $C(\Omega)$ to $C(\Omega')$. Thus a theorem of M. Stone says that $\Gamma : \mathcal{F} \to \mathcal{F}'$ is implimented by a continuous one to one and onto map $\gamma : \Omega' \to \Omega$ such that

$$\alpha(\psi)(\omega') = \psi \circ \gamma(\omega')$$

for all $\psi \in \mathcal{F}$.

Section 2 gives proofs for Theorem 1.1 and Theorem 1.2 in reverse order. Section 3 includes in particular a proof for Theorem 1.6. Thorem 1.6 in particular also solves a number of related problems in non-commutative approximation theory [Ar4], convex function theory [Pz], model theory [Ag] that William Arveson reviewed in his paper [Ar4]. We briefly indicate those related problems as well for illustration of our results and it's domain of applications.

Finally we comment now on non-separable operator systems. That there exists enough boundary representation for a non-separable operator system is proved in a recent submission to mathematics Arxiv by K. Davidson and M. Kennedy [DaK13]. Thus (a) imples (b) and (b) implies (c) goes without much change of the argument used for separable case by [Ar3]. Since Theorem 1.1 holds good for non-separable operator systems as well, it is likely that one can settle hyper-rigidity conjecture i.e. (b) implies (a) also for non-separable case provided we can avoid disintegration that is used in the last part of Theorem 1.6. We avoid many possible application as it would be marely applying our results proven here clbded with many interesting situation listed in William Arveson's paper [Ar4] and [Ar5].

I express my gratitude and thanks to Gilles Pisier and Éric Ricard for pointing out a gap in the proof of Theorem 1.2 given in the first draft with an instructive counter example which helped me to rectify the statement of Theorem 1.2 to it's present form.

2 Hann-Banach-Arveson's extension theorem:

We start with a simple lemma.

LEMMA 2.1: Let $\mathcal{S}$ be a finite dimensional operator system of a unital C^* algebra $\mathcal{A}$ and $f : \mathcal{S} \to C$ be a unital faithful positive linear functional on $\mathcal{S}$. Then for any $x \geq 0$ non zero element in $\mathcal{A}$, there exists a state g on $\mathcal{S}_x = \mathcal{S} + Cx$, extending f so that $g(x) > 0$. Further if $\mathcal{A}$ is separable then there exists a faithful state g on $\mathcal{A}$ extending f.

PROOF: For $x \in \mathcal{S}$ we take $g = f$ and extend using Krein's theorem. For $x \notin \mathcal{S}$ and $x \geq 0$, we consider the operator system $\mathcal{S}_x$ generated by $\{\mathcal{S}, x\}$. We first consider the self-adjoint subspaces $\mathcal{S}^h_x$ and $\mathcal{S}^h$ of $\mathcal{S}_x$ and $\mathcal{S}$ respectively. We recall Minkowski's function $P : \mathcal{S}^h_x \to I\!R_+$ given by

$$P(y) = \inf_{y' \in \mathcal{S}^h} \{f(y') : y \leq y'\}$$

In particular since $y \leq ||y||I$ and $f(I) = 1$, we have $P(y) \leq ||y||$ for all $y \in \mathcal{S}^h_x$. So we get

$$P(y) = \inf_{y' \in \mathcal{S}^h} \{f(y') : y \leq y' \leq ||y||I\}$$

We also have by definition that $P(y) = f(y)$ for all $y \in \mathcal{S}^h$.

f has a Hann-Banach extension as a positive linear functional g to $\mathcal{S}^h_x$ provided we choose $\alpha = g(x)$ satisfying $g(y) \leq P(y)$ for all $y \in \mathcal{S}^h_x$ i.e. by usual sub-additive property of P

$$\beta_1 = \sup_{y' \in \mathcal{S}^h} \{f(y') - P(y' - x)\} \leq \alpha \leq \beta_2 = \inf_{y \in \mathcal{S}^h} \{P(y + x) - f(y)\}$$

By our construction we have $0 \leq \beta_1 \leq \alpha \leq \beta_2$. Thus we have to rule out the possibility that $\beta_1 = \beta_2 = 0$.

Suppose that $\beta_1 = \beta_2 = 0$. So $f(y') - P(y' - x) \leq 0 \leq P(y + x) - f(y)$ for all $y', y \in \mathcal{S}^h$. Since $x \geq 0$, by definition of P we have $P(y' - x) \leq f(y')$ for all $y' \in \mathcal{S}^h$. So $f(y') = P(y' - x)$ for all $y' \in \mathcal{S}^h$. We fix $\epsilon > 0$ and choose $y \in \mathcal{S}^h$ satisfying $P(y + x) - f(y) \leq \epsilon$. We also have $P(y + x) + \epsilon \geq f(y')$ for some $y' \in \mathcal{S}^h$ with $y + x \leq y'$. So $f(y') \leq P(y + x) + \epsilon \leq f(y) + 2\epsilon$ i.e. $f(y' - y) \leq 2\epsilon$ where $0 \leq x \leq y' - y$ for some $y, y' \in \mathcal{S}^h$. Now we apply definition of P for x to get $P(x) \leq f(y' - y) \leq 2\epsilon$. Since ϵ can be arbitrarily small, we get $P(x) \leq 0$. Since f is positive on $\mathcal{S}$, we also get $P(x) \geq 0$. So we arrive finally at $P(x) = 0$.

The set $\{y' \in \mathcal{S}^h : x \leq y' \leq ||x||I\}$ is closed. If $\mathcal{S}$ is a finite dimension the closed set is also compact. In such a case we get $P(x) = f(y')$ for some $y' \in \mathcal{S}^h$ where $0 \leq x \leq y'$. This contradicts faithfulness of f on $\mathcal{S}^h_+$.

Now we extend g to $\mathcal{S}_x$ by expressing $\mathcal{S}_x = \mathcal{S}^h_x + i\mathcal{S}^h_x$ and extending linearly over the field of complex numbers.

For the last part, using Krein's theorem, we extend g to a state on $\mathcal{A}$ and denote it by g_x. Thus $g_x(x) > 0$. $\mathcal{A}$ being separable the $\mathcal{S}^1_+ = \{x \in \mathcal{S}_+ : ||x|| = 1\}$ is a Lindelöf space i.e. every open cover has a countable sub-cover. For each $x \in \mathcal{A}$, by continuity of g_x, we find an open set $\mathcal{O}_x$ in $\mathcal{S}^1_x$ such that $\phi_x(y) \geq 0$ for all $y \in \mathcal{O}_x$. Since the collection $\{\mathcal{O}_x : x \in \mathcal{S}^1_x\}$ is an open cover for $\mathcal{S}^1_+$, we get a countable sub-cover $\mathcal{O}_{x_n}$ for $\mathcal{S}^1_+$. Now we set $\phi = \sum_n \frac{1}{2^n} \phi_{x_n}$, which is a faithful state on $\mathcal{A}$. $\blacksquare$

Note that finite dimensional property of $\mathcal{S}$ is only used to ensure compactness property of the closed set $\{y' \in \mathcal{S}_h : y \leq y' \leq ||y||I\}$. Thus one simple infinite dimensional variation of Lemma 2.1 is as follows. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{S}$ is closed in weak* topology then $\{y' \in \mathcal{S}_h : y \leq y' \leq ||y||I\}$ is weak* compact. Thus by the argument used in the proof for Lemma 2.1, any weak* continuous linear functional $f : \mathcal{S} \rightarrow C$ has a faithful extension to $\mathcal{A}$ provided f is faithful on $\mathcal{S}_+$.

LEMMA 2.2: Let $\mathcal{S}$ be an operator system of a unital C^* algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{S}$ is also closed in weak* topology of $\mathcal{B}(\mathcal{H})$. Let $f : M_n(\mathcal{S}) \to C$ be a unital positive linear weak* continuous functional and be faithful on $M_n(\mathcal{S})_+$ then for any $x \geq 0$ non zero element in $\mathcal{A}$, there exists a positive linear functional $g : M_n(\mathcal{S})_x \to C$ extending f so that $g(x \otimes I_n) > 0$ where $M_n(\mathcal{S})_x = M_n(\mathcal{S}) + Cx$. Furthermore if $\mathcal{A}$ is separable then there exists a state g on the operator system $M_n(\mathcal{S}) + \mathcal{A}$ extending f which is faithful on $\mathcal{A}$. Let $\tau : \mathcal{S} \to M_n(C)$ be a UCP map such that τ is faithful on $\mathcal{S}_+$ i.e. $\tau(z) = 0$ for $z \in \mathcal{S}_+$ if and only of $z = 0$ then $\hat{s}((y^i_j) = \frac{1}{n} \sum_{1 \leq i,j \leq n} < e_i \tau(y^i_j) e_j >$ is faithful on $M_n(\mathcal{S})_+$.

PROOF: We replace the role of $\mathcal{S}$ in Lemma 2.1 by $M_n(\mathcal{S})$ and fix non zero $x \geq 0$ in $\mathcal{A}$. Assume contrary that such a state is not possible. Then going along the same line in the proof of Lemma 2.1, we get $y = (y^i_j) \in M_n(\mathcal{S})$ with $0 \leq x \otimes I \leq y$ and $f(y) = 0$. Now we consider the set $\{y' \in M_n(\mathcal{S})_+, y' \geq x \otimes I_n\}$. Since $M_n(\mathcal{S})_+$ is a closed set in weak* topology, any totally ordered set has a greatest lower bound and thus by Zorn's lemma we get a unique positive element y_{min} such that $x \otimes I \leq y_{min}$ and thus we also get $f(y_{min}) = 0$ since $y_{min} \leq y$. Since $M_n(\mathcal{S})$ is invariant under unitary conjugation by any unitary matrix $I \otimes \lambda$ where $\lambda \in M_n(C)$, we get by uniqueness y_{min} commutes with all $I \otimes \lambda$. Since in general closed convex cone generated by $\mathcal{S}_+ \otimes M_n(C)_+$ need not be equal to $M_n(\mathcal{S})_+$, we can not conclude that $y_{min} = y_0 \otimes I_n$ for some $y_0 \in \mathcal{S}$.

We will injective property of τ now. We write $y_{min} = (y^i_j)$ and

$$\sum_{i,j} < \lambda e_i, \tau(y^i_j) \lambda e_j >= 0$$

for all $\lambda \in U_n(C)$. We claim that $\tau(y^i_j) = 0$ for all $1 \leq i,j \leq n$. We fix an index $1 \leq i \leq n$. Taking $e_i \to -e_i$ and $e_j \to e_j$ for $j \neq i$, we get

$$\sum_{j \neq i} < \lambda e_i, \tau(y^i_j) \lambda e_j >= 0$$

for $\lambda \in U_n(C)$. Fix any $j \neq i$ and choose transformation $e_j \to z e_j$ where $z \in C$ and $|z| = 1$, we conclude $< \lambda e_i, \tau(y^i_j) \lambda e_j >= 0$. Since λ is any unitary matrix, $i \neq j$

we have $\tau(y^i_j) = 0$. Since $f(y) = 0$, we get $< \lambda e_i, \tau(y^i_i)\lambda e_i >= 0$ and once again λ being arbitrary unitary matrix, we conclude that $\tau(y^i_i) = 0$. Since τ is CP and faithful i.e. $\tau(z) = 0$ for $z \in \mathcal{S}_+$ if and only if $z = 0$, we conclude that $y^i_j = 0$ for all $1 \le i \le n$. Since $y = (y^i_j) \ge 0$, we conclude that $y = 0$ diagonal entries being zero. This contradicts our starting assumption that $x \ge 0$ and $x \ne 0$ since $x \otimes I_n \le y$.

Rest of the statement follows along the same line of the proof given for existence of a faithful extension of state in Lemma 2.1. ■

Let $\mathcal{S}$ be an operator system of a unital C^*-algebra $\mathcal{A}$. We recall in the following some essential steps in Hann-Banach-Arveson's extension theorem [Pa, Chapter 6]. We set one to one correspondence between the set of CP maps $\tau : \mathcal{S} \to M_n(C)$ and positive functional $s_\tau : M_n(\mathcal{S}) \to C$ defined by

$$s_\tau((x^i_j)) = \frac{1}{n} \sum_{1 \le i,j \le n} \tau(x^i_j)^i_j$$

and

$$\tau_s(x)^i_j = ns(x \otimes |e_i >< e_j|)$$

where e_i is an orthonormal basis for C^n. Note that $s_\tau(x \otimes I_n) = tr_0(\tau(x))$ for all $x \in \mathcal{S}$ where tr_0 is the normalized trace on $M_n(C)$ and further

(a) τ is unital if and only if s_τ is a state on the operator space $M_n(\mathcal{S})$;

(b) $x \to tr_0(\tau(x)$ is a state on $\mathcal{S}$;

(c) Let $S_{tr_0 \circ \tau}(\mathcal{A})$ be the set of states on $\mathcal{A}$ extending the state $x \to tr_0(\tau(x))$ from $\mathcal{S}$ to $\mathcal{A}$ i.e. $\phi \in S_{tr_0 \circ \tau}(\mathcal{A})$ if $s_\tau(x \otimes I_n) = \phi(x)$ for all $x \in \mathcal{S}$.

In order to deal with a faithful state ϕ_0 on $M_n(C)$ we fix an orthonormal basis e_i and $\lambda_i \ne 0$ such that $\phi_0(x) = \sum_i |\lambda_i|^2 < e_i, xe_i >$ for all $x \in M_n(C)$ and consider the non-negative matrix $\lambda = ((\bar{\lambda_i}\lambda_j))$ and reset

$$s_{\tau,\lambda}((x^i_j)) = s_\tau(((\lambda^i_j)) \circ ((x^i_j)))$$

$$\tau_s(x)^i_j = \frac{1}{\lambda^i_j} s_{\tau,\lambda}(x \otimes |e_i >< e_j|)$$

where $\circ$ denotes Schur product. Since Schur product takes a non-negative element to another non-negative element on $M_n(\mathcal{S})$, we inherits all the property of the correspondence between τ and s_τ with a modification $s_{\tau,\lambda}(x \otimes I_n) = \phi_0(\tau(x))$ for all $x \in \mathcal{A}$. Thus (b) is now modified as
(b') If ϕ is a state on $\mathcal{A}$ then $\phi_0(\tau(x)) = \phi(x)$ for all $x \in \mathcal{S}$ if and only if $s_{\tau,\lambda}(x \otimes I_n) = \phi(x)$ for all $x \in \mathcal{S}$. We set convex set $S_{\phi_0 \circ \tau}(\mathcal{A})$ similar to $S_{tr_0 \circ \tau}(\mathcal{A})$.

THEOREM 2.3: Let $\mathcal{S}$ be an operator system in a unital C^* algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\tau : \mathcal{S} \to M_n(C)$ be a UCP map. Let ϕ_0 be a faithful state on $M_n(C)$ and ϕ be a state on $\mathcal{A}$ such that $\phi(x) = \phi_0(\tau(x))$ for all $x \in \mathcal{S}$. If $\mathcal{S}$ is closed in weak* topology of $\mathcal{B}(\mathcal{H})$ and the map τ is injective on $\mathcal{S}$, then τ has a CP extension $\eta : \mathcal{A} \to M_n(C)$ of the $\tau : \mathcal{S} \to M_n(C)$ such that $\phi_0 \eta = \phi$ on $\mathcal{A}$.

PROOF: We consider the case first with ϕ_0, the normalized trace. We consider the linear function $\hat{s}$ on the operator system $\mathcal{S}_n$, the linear span of $M_n(\mathcal{S})$ and $\mathcal{A} \otimes I_n$ defined by

$$\hat{s}(((x^i_j)) + x \otimes I_n) = s_\tau(((x^i_j))) + \phi(x)$$

where $x^i_j \in \mathcal{S}$ and $x \in \mathcal{A}$ with $\phi \in S_\tau(\mathcal{A})$. That it is well defined follows as $s_\tau(x \otimes I) = \phi(x)$ for $x \in \mathcal{S}$. We claim that $\hat{s}$ is contractive for any faithful state $\phi \in S_\tau(\mathcal{A})$.

The linear functional being on an operator system, contractive property is equivalent to positivity of $\hat{s}$. For any element $Y = ((x^i_j)) + x \otimes I_n \geq 0$, with representing element $x^i_j \in \mathcal{S}$ and $x \in \mathcal{A}$, we assume without loss of generality that $x \in \mathcal{A}_h$ and $((x^i_j)) \in M_d(\mathcal{S})_h$ i.e. are self-adjoint elements.

We consider the maximal set $\mathcal{M}$ of self-adjoint elements in $\mathcal{A}$ such that $x \in \mathcal{M}$ for which whenever

$$0 \leq ((x^i_j)) + x \otimes I_n \tag{2.1}$$

for $(x^i_j) \in M_n(\mathcal{S})_h$ then

$$0 \leq \hat{s}((x^i_j) + x \otimes I_n) \tag{2.2}$$

Certainly $\mathcal{S}_h \in \mathcal{M}$. We aim to prove $\mathcal{M} = \mathcal{A}_h$, self-adjoint elements in $\mathcal{A}$ under our hypothesis that $\mathcal{S}$ is closed in weak* topology of $\mathcal{B}(\mathcal{H})$. Suppose not i.e. $\mathcal{M} \neq \mathcal{A}_h$. Then we find element $x \in \mathcal{A}_h$ but not in $\mathcal{S}_h$ such that

$$\hat{s}((x^i_j)) - \phi(x) < 0$$

though $(x^i_j) - x \otimes I_n \geq 0$ for some $(x^i_j) \in M_n(\mathcal{S})_h$. As in Lemma 2.2 we have $x \leq y_{min}$ where $y_{min} \in \mathcal{S}_h$ is the greatest lower bound in the weak* closed set $\{y \geq x \otimes I_n : y \in M_n(\mathcal{S})_h\}$. Thus we have $(I \otimes \lambda)y_{min}(I \otimes \lambda^*) = y_{min}$ by the uniqueness of greatest lower bound and

$$\hat{s}(y_{min}) = \hat{s}((I \otimes \lambda)y_{min}(I \otimes \lambda^*))$$

Now we follow the foot steps of the last Lemma 2.2 to conclude that $\tau(y^i_j) = 0$ for all y^i_j where $y_{min} = (y^i_j)$. τ being injective $y_{min} = 0$. So $-\phi(x) < 0$ for $-x \geq 0$. Since ϕ is state on $\mathcal{A}$, we get a contradiction.

$\hat{s}$ is a contractive unital map on an operator system. So by Krein's theorem we can extend $\hat{s}$ to $M_n(\mathcal{A})$ as a state. Thus we get a trace preserving unital extension of $\tau : \mathcal{S} \to M_n(C)$ to $\tau : \mathcal{A} \to M_n(C)$ by the above correspondence discussed preceding the statement of this theorem.

We need to include very little modification of the above argument with $s_{\tau,\lambda}$ replacing the role of s_τ in order to include more general state ϕ on $\mathcal{A}$. ■

The following theorem is little more general version of Theorem 2.3.

THEOREM 2.4: Let $\mathcal{S}$ be an operator system in a unital C^* algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H}')$ be a unital injective CP map. Let ϕ_0 be a faithful normal state on $\mathcal{B}(\mathcal{H}')$ and ϕ be a faithful state on $\mathcal{A}$ such that $\phi(x) = \phi_0(\tau(x))$ for all $x \in \mathcal{S}$. If $\mathcal{S}$ is closed in weak* topology of $\mathcal{B}(\mathcal{H})$ then τ has a CP extension $\eta : \mathcal{A} \to \mathcal{B}(\mathcal{H}')$ of $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H}')$ such that $\phi_0\eta = \phi$ on $\mathcal{A}$.

PROOF Same proof will work for a faithful normal state on $\mathcal{B}(\mathcal{H}')$ where we need to consider matrices of countable dimension $((x^i_j))$ where x^i_j are zero except finitely

many entries and Schur's product is still well defined taking product of two non-negative elements to another non-negative element of finite support. Thus one to one correspondence $\tau \to \hat{s}_{\tau,\phi_0}$ still valid with obvious modification. Rest of argument for construction of η will follow the same steps. ■

PROPOSITION 2.5: Let $\mathcal{S}$ be an operator system of a unital C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and the closed C^* algebra generated by $\mathcal{S}$ be equal to $\mathcal{A}$ i.e. $\mathcal{A} = C^*(\mathcal{S})$. Let $\tau : \mathcal{A} \to \mathcal{A}$ be a unital CP map with a faithful invariant state extending the inclusion map $I : \mathcal{S} \to \mathcal{A}$. Then τ is the identify map on $\mathcal{A}$.

PROOF: It is fairly well known that the set $\mathcal{N} = \{x \in \mathcal{A} : \tau(x) = x\}$ is a $*$-sub-algebra for a unital CP map with a faithful invariant state. Proof goes as follows. By Kadison Schwarz inequality [Ka] we have $\tau(x^*x) \geq \tau(x^*)\tau(x)$ for all $x \in \mathcal{A}$ and if equality holds for some x then we also have $\tau(x^*y) = \tau(x^*)\tau(y)$ for all $y \in \mathcal{A}$. Now we use faithfulness of the invariant state to show first that x^*x in $\mathcal{N}$ whenever $x \in \mathcal{N}$ and then $x^*y \in \mathcal{N}$ when $x, y \in \mathcal{N}$. τ being an extension of identity map on $\mathcal{S}$, it contains $\mathcal{S}$. Thus $\mathcal{N}$ also contains $*$-algebra generated by $\mathcal{S}$. τ being continuous, we conclude that $\mathcal{N} = \mathcal{A}$ i.e. τ is the identity map on $\mathcal{A}$. ■

THEOREM 2.6: Let $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ be a unital complete order isomorphism where $\mathcal{S}$ and $\mathcal{S}'$ are unital operator systems of a matrix algebra say $M_n(C)$. Further we assume complete order isomorphism takes one faithful state to another i.e. there exists faithful states ϕ, ϕ' on $M_n(C)$ such that $\phi\mathcal{I}(x) = \phi'(x)$ for all $x \in \mathcal{S}$. Then $\mathcal{I}$ has a complete order isomorphic extension to $\mathcal{I} : \mathcal{B} \to \mathcal{B}'$ preserving $\phi\mathcal{I}(x) = \phi'(x)$ for all $x \in \mathcal{B}$ where $\mathcal{B}$ and $\mathcal{B}'$ are the minimal C^* sub-algebras of $M_n(C)$ containing $\mathcal{S}$ and $\mathcal{S}'$ respectively.

PROOF: Without loss of generality we assume that $\mathcal{A}$ and $\mathcal{A}'$ are generated by $\mathcal{S}$ and $\mathcal{S}'$ respectively. Let $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{I}^{-1}}$ be Arveson's extension of the map $\mathcal{I}$ and it's inverse $\mathcal{I}^{-1}$ so that $\phi'\tau_{\mathcal{I}} = \phi$ and $\phi\tau_{\mathcal{I}^{-1}} = \phi'$ (by Theorem 2.3). Then $\tau = \tau_{\mathcal{I}^{-1}} \circ \tau_{\mathcal{I}}$ is a unital map on $\mathcal{A}$ and is an extension of the inclusion map of $\mathcal{S}$ in $\mathcal{A}$ preserving

ϕ and thus by Proposition 2.5 τ is the trivial on $\mathcal{A}$. Same is also true for $\tau_{\mathcal{I}^{-1}} \circ \tau_{\mathcal{I}}$. Thus $\tau_{\mathcal{I}^{-1}}$ is the inverse of $\tau_{\mathcal{I}}$. This completes the proof. ■

In Theorem 2.6 we imposed an artificial condition on the complete order isomorphism that it takes a faithful state to another faithful state though the proof makes use of the hypothesis. In following we now prove that it is indeed artificial for a large class of operator systems namely for separable operator systems.

LEMMA 2.7: Let $\mathcal{S}, \mathcal{S}'$ be two operator systems of separable unital C^*-algebras $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{A}' \subseteq \mathcal{B}(\mathcal{H}')$ respectively. Let $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ be a unital order isomorphism map and both $\mathcal{S}$ and $\mathcal{S}'$ be closed in weak* topologies of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}')$ respectively. Then for a given faithful state ϕ on $\mathcal{A}$ (which exists) there exists a faithful state ϕ' on $\mathcal{A}'$ so that $\phi'(x) = \phi\mathcal{I}(x)$ for all $x \in \mathcal{S}$.

PROOF: Since $\mathcal{A}$ is separable, any open cover of $S^1_+ = \{x \geq 0 : ||x|| = 1\}$ has a countable sub-cover. Thus we can follow the steps given in Lemma 2.1 with $\mathcal{S} = \{I\}$ to guarantee existence of a faithful state after fixing a countable sub-cover $\{\mathcal{O}_x : x \in\}$ for S^1_+ and then by taking their convex combination $\phi = \sum_{n \geq 1} \frac{1}{2^n} \phi_{x_n}$. Now we consider the state $\phi\mathcal{I}$ on $\mathcal{S}$ and follow the same steps to find a faithful state ϕ' on $\mathcal{A}$ such that $\phi' = \phi\mathcal{I}$ on $\mathcal{S}$. ■

Given a state ϕ on a C^*-algebra $\mathcal{A}$, we consider the GNS space $(\mathcal{H}_\phi, \pi_\phi, \Omega_\phi)$ associated with $(\mathcal{A}, \phi)$ where $\phi(x) =< \Omega_\phi, \pi(x)\Omega_\phi >$ has a normal state extension to the von-Neumann algebra $\pi_\phi(\mathcal{A})''$ given by $\phi_\Omega(X) =< \Omega, X\Omega >$ for all $X \in \pi_\phi(\mathcal{A})''$ where $\Omega \in \mathcal{H}_\phi$ is the cyclic vector for $\pi(\mathcal{A})''$ in $\mathcal{H}_\phi$. For a faithful state ϕ on $\mathcal{A}$, vector state ϕ_Ω is faithful on $\pi_\phi(\mathcal{A})''$.

THEOREM 2.8: Any complete order isomorphism between two operator systems of $M_n(C)$ is implemented by some unitary matrix.

PROOF: It is a simple consequence now once we club Lemma 2.7 and Theorem 2.6 that any complete order isomorphism has an extension to a complete order iso-

morphism between there C^* algebras generated by them. Thus the result follows by a simple application of Corollary 5.2.3 in [ER] which says any complete order isomorphism between two C^*-algebras are C^*-isomorphism. ■

PROOF OF THEOREM 1.1: Proof goes qualitatively as in Theorem 2.8 aided with Theorem 2.3. Let $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$ be a complete order isomorphism. If $\mathcal{S}$ and so $\mathcal{S}'$ are finite dimensional operator systems, we fix faithful states ϕ and ϕ' on $C^*(\mathcal{S})$ and $C^*(\mathcal{S}')$ respectively with the property $\phi' = \phi \circ \mathcal{I}^{-1}$ using separable property of $C^*(\mathcal{S})$ and $C^*(\mathcal{S}')$ and argument that we used in the proof for Lemma 2.1. Thus we extend a C^* isomorphism going along the line of Theorem 2.8.

For separable but not finite dimensional operator systems $\mathcal{S}$ and $\mathcal{S}'$, we consider the set of finite dimensional operator subsystems $\mathcal{S}(f)$ of $\mathcal{S}$ i.e. $\mathcal{S}(f) = \{\mathcal{S}_f \subseteq \mathcal{S} : \mathcal{S}_f$ operator systems of $\mathcal{A}\}$. We fix an increasing chain of $\mathcal{S}_{f_n}$ of operator systems such that $C^*(\bigcup \mathcal{S}_{f_n}) = C^*(\mathcal{S})$ and $f_{n+1} \notin C^*(\mathcal{S}_{f_n})$ since $\mathcal{S}$ is separable. Since $C^*(\bigcup \mathcal{S}_{\mathcal{F}_n}) = C^*(\mathcal{S})$, without loos of generality we assume that $\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}_{f_n}$ and $\mathcal{S}' = \bigcup_{n \geq 1} \mathcal{I}(\mathcal{S}_{f_n}))$.

Thus we find an isomorphism $\mathcal{I} : C^*(\mathcal{S}) \to C^*(\mathcal{S}')$ by extending $\mathcal{I} : C^*(\mathcal{S}_{f_n}) \to C^*(\mathcal{I}(\mathcal{S}_{f_n}))$ to $\bigcup_{n \geq 1} C^*(\mathcal{S}_{f_n}) \to \bigcup_{n \geq 1} C^*(\mathcal{I}(\mathcal{S}_{f_n}))$ and then to their closures.

Now we extend the result to non-separable operator systems. We consider the set of separable operator subsystems $\mathcal{S}_S$ in $\mathcal{S}$ and $\mathcal{S}'_S$ in $\mathcal{S}'$ respectively. We fix a complete order isomorphism $\mathcal{I} : \mathcal{S} \to \mathcal{S}'$. For each element $\mathcal{S}_s \in \mathcal{S}_S$, we get an isomorphism $\mathcal{I}_S : C^*(\mathcal{S}_s) \to C^*(\mathcal{I}(\mathcal{S}_s))$. We use now transfinite induction to get an isomorphism $\mathcal{I} : \bigcup_{\mathcal{S}_s \in \mathcal{S}_S} C^*(\mathcal{S}_s) \to \bigcup_{\mathcal{S}'_s \in \mathcal{S}'_S} C^*(\mathcal{S}'_s)$ and once more we use dense property to get an unique isomorphism as an extension to $C^*(\mathcal{S})$ with $C^*(\mathcal{S}')$. ■

A subspace $\mathcal{M}$ of a unital C^*-algebra $\mathcal{A}$ is called operator space. We denote by $C^*(\mathcal{M})$ the C^*-algebra generated by $\mathcal{M}$ together with $\mathcal{M}^* = \{x : x^* \in \mathcal{M}\}$. Next result answers when $\mathcal{A}(x)$ and $\mathcal{A}(y)$ are expected to be C^*-isomorphic under the natural map $\lambda + \mu x \to \lambda + \mu y$ for two elements $x, y \in \mathcal{A}$.

THEOREM 2.9: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two operator space with units of unital C^* algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. If $\mathcal{I}_0 : \mathcal{M}_1 \to \mathcal{M}_2$ is a unital complete isometric map then there exists a unique extension of $\mathcal{I}_0$ to a C^*-isomorphism $\mathcal{I} : C^*(\mathcal{M}_1) \to C^*(\mathcal{M}_2)$.

PROOF: We consider the map $\hat{\mathcal{I}}_0 : \mathcal{M}_1 + \mathcal{M}_1^* \to \mathcal{M}_2 + \mathcal{M}_2^*$ defined by $\hat{\mathcal{I}}_0(a+b) = \mathcal{I}_0(a) + \mathcal{I}_0(b)$. By Proposition 2.12 in [Pa2], $\hat{\mathcal{I}}_0$ is a well defined positive map since $\mathcal{I}_0$ is contractive map. Same argument now also shows that the map $\hat{\mathcal{I}}_0 \otimes I_n$ is a well defined positive map for each $n \geq 0$ since $\mathcal{I}_0 \otimes I_n$ is contractive. Same argument holds good for the map $\hat{\mathcal{J}}_0 : \mathcal{M}_2 + \mathcal{M}_2^* \to \mathcal{M}_1 + \mathcal{M}_1^*$ defined by

$$\hat{\mathcal{J}}_0(a+b) = \mathcal{I}_0^{-1}(a) + \mathcal{I}_0^{-1}(b)$$

for all $a, b \in \mathcal{M}_2$. Clearly $\hat{\mathcal{J}}_0$ is the inverse of $\hat{\mathcal{I}}_0$. Thus $\hat{\mathcal{I}}_0$ extends uniquely to a isomorphism of their C^*-algebras by Theorem 1.1. ■

3 Non-commutative Choquet's boundary theory and Hyper-rigidity:

Proof of Theorem 1.6 relies on various inputs apart from Theorem 1.1. In particular we make use of Theorem 2.5 in [Ar3] which is a refined version of a result of M. Dritschel and S. McCullough [DM]. In the following we briefly recall these inputs first.

Given an operator system $\mathcal{S} \subseteq C^*(\mathcal{S})$ and two UCP maps $\tau_k : \mathcal{S} \to \mathcal{B}(\mathcal{H}_k)$, $k = 1, 2$ we write $\tau_1 \preceq \tau_2$ if $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and $P_{\mathcal{H}_1} \tau_2(a) P_{\mathcal{H}_1} = \tau_1(a)$ for all $a \in \mathcal{S}$. The *partial ordering* is *reflexive* and *transitive* . Any UCP map can be dilated in a trivial way by taking direct sum with another UCP map.

DEFINITION 3.1: A UCP map $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is said to be *maximal* if it has no non-trivial dilations: $\tau \preceq \tau'$ implies that $\tau' = \tau \oplus \psi$ for some unital CP map $\psi : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}')$.

A characterization of the boundary representations due to P. S. Muhly and Baruch Solel [MuS] found it's use in proving Theorem 2.5 in Arveson's recent paper [Ar3]. In the following we adopt for our purpose the following two propositions.

PROPOSITION 3.2: [MuS] A UCP map $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ has the unique extension property if and only if τ is maximal.

PROOF: We refer to Proposition 2.4 in [Ar3] for a proof. ■

PROPOSITION 3.3: Let $\mathcal{S} \subseteq C^*(\mathcal{S})$ be a separable operator system and $\tau_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a separably acting UCP map. Then τ_0 has a dilation to a separable acting UCP map $\tilde{\tau}_0 : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}})$ with the unique extension property i.e. there exists a unique UCP extension $\tilde{\tau} : C^*(\mathcal{S}) \to \mathcal{B}(\tilde{\mathcal{H}})$ of $\tilde{\tau}_0 : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}})$ such that $\tilde{\tau}$ is a representation of $C^*(\mathcal{S})$ on $\tilde{\mathcal{H}}$. If P is the projection on $\tilde{\mathcal{H}}$ with range equal to $\mathcal{H}$ then UCP map

$$\tau(a) = P\tilde{\tau}(a)P$$

for all $a \in C^*(\mathcal{S})$ extends UCP map $\tau_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H})$.

PROOF: This is exactly the statement of Theorem 2.5 in [Ar3]. This is a non-trivial variation of Stinespring dilation modulo maximal property and the proof given in [Ar3] uses separable property of $\mathcal{S}$ and $\mathcal{H}$. ■

We make the following two observation that we will use frequently:

(a) $\tau(a) = P\eta(a)P$ whenever $\tilde{\tau} \preceq \eta$.

(b) The extension $\tau : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ of τ_0 depends on the choice that we make to construct $\tilde{\tau}_0$. Let $\tilde{\tau}_0^k : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}}_k)$ be two such choices as extensions of τ_0. Maximal property will ensure that there exists an unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $UP_1U^* = P_2$ and

$$\tilde{\tau}_1(a) = U^*\tilde{\tau}_2(a)U$$

for all $a \in C^*(\mathcal{S})$, where $P_k : \tilde{\mathcal{H}}_k \to \mathcal{H}$ are the projections with range equal to $\mathcal{H}$ from $\mathcal{H}_k$. By looking at corner determined by P_2 we arive at $\tau_1(a) = u\tau_2(a)u^*$ for all

$a \in C^*(\mathcal{S})$. Since $\tau_1(a) = \tau_2(a) = \tau_0(a)$, we arrive at u commutes with $\tau_0(a) : a \in \mathcal{S}$. If $\tau_0(a) = \pi_0(a)$ and $\pi_0 : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ is a representation then $u \in \pi_0(C^*(\mathcal{S}))'$.

LEMMA 3.4: $\mathcal{S}$ is hyper-rigid for $C^*(\mathcal{S})$ if and only if every faithful cyclic representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ has a unique CP extension i.e. only UCP map $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that satisfies $\tau_{|}\mathcal{S} = \pi_{|}\mathcal{S}$ is $\tau = \pi$ itself.

PROOF: It is good enough if we verify unique extension property for all cyclic and faithful representation $\pi : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$ i.e. we can as well put restriction in the definition 1.4 that π is any cyclic and faithful representation. We may take π to be a cyclic representation. Otherwise we decompose it into countable direct sum of cyclic representations. It is simple to check aided with Proposition 4.4 in [Ar4] that unique extension property holds for the direct sum once same holds in each copies. Further we may assume π to be also faithful. Otherwise we fix a faithful cyclic representation of $\pi_0 : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}_0)$ with cyclic vector Ω_0 and consider faithful cyclic representation $a \to \pi_0(a) \oplus \pi(a)$ in $[\pi_0 \oplus \pi(C^*(\mathcal{S}))\Omega \oplus \Omega_0] \subseteq \mathcal{H}_0 \oplus \mathcal{H}$ for an arbitrary (a non faithful) cyclic representation $\pi : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ with cyclic vector Ω and a UCP map τ such that $\tau_{|\mathcal{S}} = \pi_{|\mathcal{S}}$. Cyclic property of modified representation follows by our construction. Faithful property follows from faithful property of π_0 as follows: If $\pi_0(a) \oplus \pi(a) = 0$, we get

$$< f \oplus 0, \pi_0(a) \oplus \pi(a)\pi_0(b) \oplus \pi(b)\Omega_0 \oplus \Omega >= 0$$

for all $f \in [\pi_0(C^*(\mathcal{S}))\Omega_0] = \mathcal{H}_0$ i.e. $< f, \pi_0(a)\pi_0(b)\Omega_0 >= 0$ and thus $\pi_0(a) = 0$. Note that UCP map $a \to \pi_0(a) \oplus \tau(a)$ agrees with $\pi_0 \oplus \pi$ on $\mathcal{S}$. Thus by our assumption we have the map $x \to \pi_0(a) \oplus \tau(a) = \pi_0(a) \oplus \pi(a)$ for all $a \in C^*(\mathcal{S})$. Thus $\pi(a) = \tau(a)$. ∎

PROPOSITION 3.5: Let $\pi : C^*(\mathcal{S}) \to \mathcal{H}(\mathcal{H})$ be a faithful representation and $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a UCP map such that $\tau = \pi$ on $\mathcal{S}$. Let $\tilde{\pi} : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}})$ be any dilation of $\tau : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ with unique extension property as described in Proposition 3.3. Then $\tilde{\pi} : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}})$ is a complete order isomorphism of $\mathcal{S}$ with it's range $\tilde{\pi}(\mathcal{S})$.

Further we have a unique extension of complete order isomorphism $\mathcal{I}_0 : \pi(\mathcal{S}) \to \tilde{\pi}(\mathcal{S})$ to a C^* isomorphism $\mathcal{I} : \pi(C^*(\mathcal{S})) \to \tilde{\pi}(\mathcal{S})$ defined by

$$\mathcal{I}(\pi(a)) = \tilde{\pi}(a) \tag{3.1}$$

for all $a \in C^*(\mathcal{S})$.

PROOF: The map $\pi : \mathcal{S} \to \pi(\mathcal{S})$ is a complete order-isomorphism since π is a faithful representation of $C^*(\mathcal{S})$. We claim that the map $\tilde{\pi} : \mathcal{S} \to \tilde{\pi}(\mathcal{S})$ is also a complete order isomorphism. Complete positive property of $\tilde{\pi}$ follows as $\tilde{\pi}$ is a restriction of a $*$-representation of $C^*(\mathcal{S})$. For one to one property of $\tilde{\pi}$ on $\mathcal{S}$ we verify that if some $a \in \mathcal{S}$, $\tilde{\pi}(a) = 0$ then $\tau(a) = P\tilde{\pi}(a)P = 0$. However $\tau = \pi$ on $\mathcal{S}$ and thus $\pi(a) = 0$. Since π is injective, we get $a = 0$. Thus inverse of $\tilde{\pi}$ which takes $\tilde{\pi}(s) \to s$ is a well defined linear map. That the inverse is also completely positive follows along the same line: Let $\tilde{\pi} \otimes I_n((s^i_j)) \geq 0$ for some $(s^i_j) \in M_n(\mathcal{S})$. Injective property of $\tilde{\pi}$ will ensure that $(s^i_j) \in M_n(\mathcal{S})$. Since $\pi(s) = \tau(s) = P\tilde{\pi}(s)P$ for all $s \in \mathcal{S}$, we also have $(\pi(s^i_j)) = (\tau(s^i_j)) \geq 0$. Since $\pi \otimes I_n$ is an order isomorphism, we get $(s^i_j) \geq 0$. Thus we get a complete order isomorphism map $\mathcal{I}_0 : \pi(\mathcal{S}) \to \tilde{\pi}(\mathcal{S})$ defined by

$$\mathcal{I}_0(\pi(a)) = \tilde{\pi}(a)$$

for all $a \in \mathcal{S}$. The last statement is a simple consequence of Theorem 1.1. ■

PROPOSITION 3.6: Let $\pi_0 : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}_0)$ be a separably acting representation of $C^*(\mathcal{S})$ and CP_{π_0} be set defined by

$$CP_{\pi_0} = \{\tau : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}) \text{ UCP map, } \tau_{|}\mathcal{S} = \pi_{|}\mathcal{S}\}$$

Then $\tau(C^*(\mathcal{S}))' = \pi_0(C^*(\mathcal{S}))'$ for all $\tau \in CP_{\pi_0}$.

PROOF: CP_{π_0} is a convex set. It is a compact set in Bounded Weak topology [Ar1,Pa2].

Let τ^1_0 be an extreme point in CP_{π_0}. We fix a dilation $\tilde{\pi}^1_0 : C^*(\mathcal{S}) \to \mathcal{B}(\tilde{\mathcal{H}}^1_0)$ of $\tau^1_0 : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}_0)$ with unique maximal extension property as described in

Proposition 3.3 by taking $\mathcal{S}$ as $C^*(\mathcal{S})$ in the proposition so that

$$\tau_0^1(a) = P_0\tilde{\pi}_0^1(a)P_0 \tag{3.2}$$

for all $a \in C^*(\mathcal{S})$ where P_0 is the projection in $\tilde{\mathcal{H}}_0^1$ with range equal to $\mathcal{H}_0 \subseteq \tilde{\mathcal{H}}_0^1$. Note that if $\tilde{\pi}_0^1 : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}}_0^1)$ is also a dilation of $\pi_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ with multiplicative property and so there exists a seprable Hilbert space $\tilde{\mathcal{H}}_0^2 \supseteq \mathcal{H}_0$ and unitary operator $V : \tilde{\mathcal{H}}_0^1 \to \tilde{\mathcal{H}}_0^2$ such that

(i) $P_0 V = V P_0$;

(ii) The UCP map $\tilde{\pi}_0^2 : a \to V^*\tilde{\pi}_0^1(a)V$ is a dilation of $\pi_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H}_0)$ and $\tilde{\pi}_0 \preceq \tilde{\pi}_0^2$ where $\tilde{\pi}_0 : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}}_0)$ is a dilation of $\pi_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H}_0)$ with unique maximal extension property. Since $\tilde{\pi}_0^2 = \tilde{\pi}_0 \oplus \eta$, if we set UCP map

$$\tau_0(a) = P_0\tilde{\pi}_2(a)P_0$$

then we still get

$$\tau^1(a) = u_1\tau_0(a)u_1^*$$

for some $u_1 \in \pi_0(C^*(\mathcal{S}))'$. In particular τ_0 is also an extreme point in CP_{π_0}.

For an arbitrary choice of dilation $\tilde{\pi} : \mathcal{S} \to \mathcal{B}(\tilde{\mathcal{H}})$ of $\tau_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H}_0)$ with unique maximal extension property we get

$$\tau(a) = P\tilde{\pi}(a)P$$

where $P : \tilde{\mathcal{H}} \to \mathcal{H}_0$ is the projection with range equal to $\mathcal{H}_0$ and

$$\tau(a) = u\tau_0(a)u^*$$

for some $u \in \pi_0(C^*(\mathcal{S}))'$ depending on the choice of $\tilde{\pi}_0$. Conversely for an element $u \in \pi_0(C^*(\mathcal{S}))'$, we can choose any unitary operator U on $\tilde{\mathcal{H}}_0$ so that $P_0U = uP_0$ on $\tilde{\mathcal{H}}_0$ and set $\tilde{\pi}_0^U(a) = U\tilde{\pi}_0(a)U^*$ and check easily that corner UCP map is given by $\tau_0^u(a) = u\tau_0(a)u^*$ for $a \in C^*(\mathcal{S})$.

Now consider the UCP map $\eta_0 : C^*(\mathcal{S}) \to \mathcal{B}(\tilde{\mathcal{H}}_0)$ defined by

$$\eta_0(a) = \frac{1}{2}(\tilde{\pi}_0(a) + \tilde{\pi}_0^U(a))$$

for $a \in C^*(\mathcal{S})$ where U is an arbitray but fixed unitary operator on $\tilde{\mathcal{H}}_0$ such that $P_0 U = uP_0$, u unitary operator on $\mathcal{H}_0$. We note that $P_0\eta_0(a)P_0 = \pi_0(a)$ if $a \in \mathcal{S}$ since u commutes with $\pi(\mathcal{S})$ by our choice.

We consider a dilation of $\eta_0 : C^*(\mathcal{S}) \to \mathcal{B}(\tilde{\mathcal{H}}_0)$ as given in Proposition 3.3 to $\tilde{\eta}_0 : C^*(\mathcal{S}) \to \mathcal{B}(\tilde{\tilde{\mathcal{H}}}_0)$ with unique maximal extension property (take $\mathcal{S} = C^*(\mathcal{S})$) in the Proposition 3.3). The map $\tilde{\eta}_0 : \mathcal{S} \to \mathcal{B}(\tilde{\tilde{\mathcal{H}}})$ is a complete order isomorphism as proven above in Proposition 3.6 for any dilation with unique extenstion property as π_0 is faithful and thus complete order isomorphism from $\mathcal{S}$ to it's range. Note that $\tilde{\eta}_0$ is also a dilation of $\pi_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H}_0)$ as $P_0\tilde{\eta}_0(a)P_0 = P_0\tilde{P}\tilde{\eta}_0(a)\tilde{P}P_0 = P_0\eta_0(a)P_0 = \pi_0(a)$ for all $a \in \mathcal{S}$.

As before we find $P_0\tilde{\eta}_0(a)P_0 = w\tau_0(a)w^*$ for all $a \in C^*(\mathcal{S})$ for some unitary $w \in \pi_0(C^*(\mathcal{S})'$. Thus we arrive at the following equality

$$w\tau_0(a)w^* = \frac{1}{2}(\tau_0(a) + u\tau_0(a)u^*)$$

for all $a \in C^*(\mathcal{S})$. Since τ_0 is an extreme point in CP_{π_0} we get $u\tau_0(a)u^* = \tau_0(a)$ for all $a \in C^*(\mathcal{S})$. This shows that $\tau_1(C^*(\mathcal{S}))' = \pi_0(C^*(\mathcal{S}))'$. Since it holds for any extreme elements in CP_{π_0}, we conclude same holds for any of their elements in their closures. Hence by Krein-Millman's theorem we conclude the proof. ■

PROOF OF THEOREM 1.6: A proof for $(a) \Rightarrow (b) \Rightarrow (c)$ is given by William Arveson in Corollary 4.2 in [Ar4]. Now in the next paragraph we will prove $(b) \Rightarrow (a)$.

We fix π, τ as in Proposition 3.7. We fix a maximal abelian von-Neumann subalgebra $\mathcal{M}$ of $\pi(C^*(\mathcal{S}))'$. $\mathcal{S}$ being separable, $C^*(\mathcal{S})$ is also separable as a C^*-algebra. We identify $\mathcal{M} = L^\infty(X, \mu)$ for a standard Borel space with a probability measure μ and decompose $\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x)$ and any element $A \in \mathcal{M}'$ as $A = \int_X^\oplus A(x)d\mu(x)$. In particular decompose π into a direct integral of Borel measurable irreducible representations [Di,BRI] $\pi_x : C^*(\mathcal{S}) \to \mathcal{B}(\mathcal{H}_x)$ given by

$$\pi(a) = \int^\oplus \pi_x(a)d\mu(x)$$

and also decompose $\tau(a)$ into a direct integral of UCP maps $\tau_x : C^*(\mathcal{S}) \to \mathcal{B}(\tilde{\mathcal{H}}_x)$ given by

$$\tau(a) = \int_X^{\oplus} \tau_x(a) d\mu(x)$$

Since $\tau(a) = \pi(a)$ for all $a \in \mathcal{S}$, we find a μ-null set N such that $\pi_x(a) = \tau_x(a)$ for all $a \in \mathcal{S}$ and $x \notin N$. Thus by (b) we get $\pi_x(a) = \tau_x(a)$ for all $a \in C^*(\mathcal{S})$ and $x \notin N$. So in particular $\pi = \tau$ on $C^*(\mathcal{S})$. That (b) implies (a) follows now by Lemma 3.4 ■

Now we aim to prove a result which we may call non-commutative analogue of potential theorem giving a criteria for hyper-rigidity of $\mathcal{S}$ for $C^*(\mathcal{S})$. Let $\pi, \tau : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be elements as described in Lemma 3.4. The unit vector Ω determines faithful state ϕ of $C^*(\mathcal{S})$ given by

$$\phi(a) =< \Omega, \pi(a)\Omega >$$

We also set a state $\tilde{\phi}$ of $C^*(\mathcal{S})$ defined by

$$\tilde{\phi}(a) =< \Omega, \tilde{\pi}(a)\Omega > \tag{3.3}$$

Thus $\tilde{\phi}(a) =< P\Omega, \tilde{\pi}(a)P\Omega >=< \Omega, \tau(a)\Omega >$ for all $a \in C^*(\mathcal{S})$.

LEMMA 3.7: Then $\tilde{\phi} = \phi$ on $C^*(\mathcal{S})$ if and only if the state $a \to< \Omega, \tau(a)\Omega >$ is faithful. In such a case $\phi(a) =< \Omega, \tau(a)\Omega >$ for all $a \in C^*(\mathcal{S})$.

PROOF: If part of the statement is trivial. For only if part we check that state $\tilde{\phi}$ being faithful on $C^*(\mathcal{S})$ and complete order isomorphism $\mathcal{I}_0 : \pi(\mathcal{S}) \to \tilde{\pi}(\mathcal{S})$ which admits $\phi\mathcal{I}_0\pi = \tilde{\phi}\tilde{\pi}$ on $\mathcal{S}$ we get same relation valid for unique C^*-isomorphic extensions by Theorem 1.1. ■

THEOREM 3.8: $\mathcal{S}$ is hyper-rigid for $C^*(\mathcal{S})$ if and only if every faithful cyclic representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ admits the following criteria: any UCP map $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that satisfies $\tau_|\mathcal{S} = \pi_|\mathcal{S}$ is faithful on $C^*(\mathcal{S})_+$ i.e. $\tau(a) = 0$ for $a \in C^*(\mathcal{S})_+$ if and only if $a = 0$.

PROOF: The proof that follows now is somewhat routine. We need to prove if part. By our hypothesis and Lemma 3.7 we have $\tilde{\phi} = \phi$ on $C^*(\mathcal{S})$. For each $a \in C^*(\mathcal{S})$ we have $\tau(a) \in \pi(C^*(\mathcal{S})''$ by Proposition 3.6 and the vector state $\phi(X) =< \Omega, X\Omega >$ is faithful on $\pi(C^*(\mathcal{S}))''$. For any $a \in \mathcal{A}$, Kadison's inequality says that $\tau(a^*a) - \tau(a^*)\tau(a) \geq 0$. Since $< \Omega, \tau(a^*a) - \tau(a)\tau(a)\Omega >= \tilde{\phi}(a^*a) - \phi(a^*a) = 0$ for all $a \in \mathcal{S}$, faithful property of that vector state says that $\tau(a^*a) = \pi(a^*a)$ for all $a \in \mathcal{S}$. Use polarization identity to conclude that $\tau(a^*b) = \pi(a^*b)$ for all $a, b \in \mathcal{S}$. Since we $\mathcal{S}^* = \mathcal{S}$, we get $\tau(a) = \pi(a)$ for any $a = a_1a_2$ with $a_1, a_2 \in \mathcal{S}$. Now we can repeat the process to show $\tau(a) = \pi(a)$ for all $a = a_1..a_n$ with $a_k \in \mathcal{S}$. This shows that $\tau(a) = \pi(a)$ for all a in the dense $*$-algebra of $C^*(\mathcal{S})$. Thus result follows since both the maps are contraction being positive unital maps on a C^*-algebra. ■

Now we take a break from separablity hypothesis on $\mathcal{S}$ and $\mathcal{H}$ in our formalism for hyper-rigidity. Though Proposition 3.3 uses sepability of $\mathcal{S}$ and $\mathcal{H}$, one can use transfinite induction to arrive at a result for non-separable $\mathcal{S}$ and $\mathcal{H}$ as well. Proof follows standard method by considering all separable operator sub-systems in $\mathcal{S}$ and separable subspace of $\mathcal{H}$. For an UCP map $\tau_0 : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ in CP^{π_0} we may consider the family of UCP maps $\tau_0' : \mathcal{S}' \to \mathcal{B}(\mathcal{H}')$ defined by restricting the map τ_0 to $\mathcal{S}'$ and then lookig at the corner by contraction with the projections with range of countable dimension. Let $\tilde{\tau}_0' : \mathcal{S}' \to \mathcal{B}(\tilde{\mathcal{H}}')$ be a separable dilation of τ_0' with maximal extension property given in Proposition 3.3. Two such dilations are identified as one can be obtained by conjugating the other and thus such a dilation with maximal extension property is unique modulo a unitary conjugation.

We set a partial ordering on the collections of such dilations by declaring $\tilde{\tau}_0' \preceq \tilde{\tau}_0''$ if $\mathcal{S}' \subseteq \mathcal{S}''$ and $\mathcal{H}' \subseteq \mathcal{H}$. We take Hilbert space $\mathcal{H}$ to be the inductive limit Hilbert spaces with directed limit with inclusion. For any totally ordered subset say $\tilde{\tau}_O^\alpha : \mathcal{S} \to \mathcal{B}(\mathcal{H})$, we use compactness of UCP maps in the Bounded Weak topology, to find a limit point of $\eta : a \to P_\alpha \tilde{\tau}_0^\alpha(a) P_\alpha,\ a \in C^*(\mathcal{S})$ as $\mathcal{H}_\alpha \uparrow \mathcal{H}$. The limit point is a UCP map and $P_0 \eta_{\mathcal{S}}(a) P_0 = \pi_0(a)$ for $a \in \mathcal{S}$. Since Since the map $\eta_\alpha : a \to P_\alpha \eta(a) P_\alpha$

is a $*$-homomorphism on $C^*(\mathcal{S}_\alpha)$, consistnacy condition ensures that η is also a $*-$homomorphism. Thus Zorn Lemma ensures existence of an maximal element in the partially ordered set and two such maximal elements are unitary conjugate to each other.

Thus in the proof of Theorem 1.6, we used separability of $\mathcal{S}$ only in the last leg of the argument to disintegrate π. However note that a faithful representation of $C^*(\mathcal{S})$ is needed to ensure Lemma 3.4 and Proposition 3.5 to be valid. In such a case Theorem 3.8 is also valid. In particular C^* algebra $\mathcal{B}(\mathcal{H})$, which is not a separable C^*-algebra for an infinite dimensional separable Hilbert space $\mathcal{H}$, does fall in this catagory. More generally any $\sigma-$finite von-Neumann algebra falls in the category of C^*-algebra that admits a faithful (normal) state. We sum up now saying that Theorem 3.8 is valid for a non-separable operator system $\mathcal{S}$ provided $C^*(\mathcal{S})$ admits a faithful representation. Statements (a) implies (b) and (b) implies (c) in Theorem 1,6 follows along the same line for non-separable operator systems once we make use recent announced results on existence of sufficiently many boundary representation [DaK] even for non-separable operator system $\mathcal{S}$. It is not clear how to show (b) implies (a) which is likely to be true if we go by general wisdom!

REFERENCES

- [Ag] Agler, J.: Surveys of some recent results in operator theory, Vol. II, 123, Longman Sci. Tech., Harlow, 1988
- [Ar1] Arveson, William B.: Sub-algebras of C^*-algebras, Acta. Amth. 123 (1969) 141-224.
- [Ar2] Arveson, William B. Sub-algebras of C^*-algebras. II. Acta Math. 128 (1972), no. 3-4, 271-308.
- [Ar3] Arveson, William B.: The non-commutative Choquet boundary. J. Amer. Math. Soc. 21 (2008), no. 4, 1065-1084.

- [Ar5] Arveson, William B..: The non-commutative Choquet boundary II: hyper-rigidity. Israel J. Math. 184 (2011), 349385.
- [Ar4] Arveson, William B.: The non-commutative Choquet boundary III: operator systems in matrix algebras. Math. Scand. 106 (2010), no. 2, 196210.
- [Bl] Blecher, D.: The Shilov boundary of an operator space and the characterization theo- rems. J. Funct. Anal., 182:280343, 2001.
- [BlM] Blecher, D. , Merdy, C. Le : Operator algebras and their modules, volume 30 of LMS Monographs. Clarendon Press, Oxford, 2004.
- [BR] Bratteli, Ola., Robinson, D.W. : Operator algebras and quantum statistical mechanics, I, Springer 1981.
- [Ch] Choi, M.D.: Completely positive linear maps on complex matrices, Linear Algebra and App (10) 1975 285-290.
- [Da] Davidson, Kenneth R.: The mathematical legacy of William Arveson. (English summary) J. Operator Theory 68 (2012), no. 2, 307-334. 1841-7744.
- [DaK] Davidson, Kenneth R., Kennedy, M.: The Choquet boundary of an operator system, Arxiv:1303.3252, 2013.
- [Dix] Dixmier, J.: C*-algebras. Translated from the French by Francis Jellett. North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
- [DM] Dritschel, Michael A. McCullough, Scott A. Boundary representations for families of representations of operator algebras and spaces. J. Operator Theory 53 (2005), no. 1, 159-167.
- [ER] Effros, Edward G.; Ruan, Zhong-Jin: Operator spaces. London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000.

- [Fa] Farenick, Douglas; Arveson's criterion for unitary similarity. Linear Algebra Appl. 435, no. 4, 769-777 (2011).
- [Ha1] Hamana, M.: Injective envelopes of C -algebras. J. Math. Soc. Japan, 31:181-197, (1979).
- [Ha2] Hamana, M.: Injective envelopes of operator systems. Publ. Res. Inst. Math. Sci. (Kyoto), 15: 773-785, (1979).
- [Ka1] Kadison, Richard V.: Isometries of operator algebras, Ann. Math. 54(2)(1951) 325-338.
- [Ka2] Kadison, Richard V.: A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. of Math. (2) 56, 494-503 (1952).
- [Ko] Korovkin,P.P.: On convergence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR (N.S.):961-964. (1953).
- [Mo] Mohari, A.: G. Birkhoff's problem in irreversible quantum dynamics arXiv:1006.5198.
- [MuS] Muhly, Paul S. ; Solel, Baruch, An algebraic characterization of boundary representations. Non-selfadjoint operator algebras, operator theory, and related topics, 189196, Oper. Theory Adv. Appl., 104, Birkhuser, Basel, 1998.
- [Pa1] Paulsen, Vern I: Completely bounded homomorphisms of operator algebras. Proc. Amer. Math. Soc. 92 (1984), no. 2, 225-228.
- [Pa2] Paulsen, Vern I: Every completely polynomially bounded operator is similar to a contraction. J. Funct. Anal. 55 (1984), no. 1, 1-17.
- [Pa3] Paulsen, V.: Completely bounded maps and operator algebras, Cambridge Studies in Advance Mathematics 78, Cambridge University Press. 2002
- [Pz] Petz, D. On the equality in Jensen's inequality for operator convex functions, Int. Eg. Op. Th., vol-9 (1986).

- [Phe] Phelps, R.R.: Lectures on Choquet's theorem, vol-1757 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, Second edition (2001).
- [Pi] Pisier, G.: Introduction to operator space theory. London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003. viii+478 pp.
- [St] Stinespring, W. F.: Positive functions on C^* algebras, Proc. Amer. Math. Soc. 6 (1955) 211-216.
- [Sto1] Stone, M. H.: The theory of representations for Boolean algebras. Trans. Amer. Math. Soc. 40 (1936), no. 1, 37-111.
- [Sto2] Stone, M. H.: Applications of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc. 41 (1937), no. 3, 375-481.